Lines in metric spaces

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In a metric space with distance function d, a point v is said to lie between points u and w if and only if $d(u, v) + d(v, w) = d(u, w)$. This notion allows us to define lines in metric spaces in different ways that generalize the standard definition of a line in the Euclidean plane. With one of these definitions, the speaker conjectured and Xiaomin Chen proved that in every metric space on n points, some line consists either of all n points or of precisely two points. (In the Euclidean plane, this statement reduces to the Sylvester-Gallai theorem.) With another definition, Chen and the speaker conjectured that in every metric space on n points, some line consists of all n points or else there are at least n distinct lines. (In the Euclidean plane, this statement reduces to a corollary of the Sylvester-Gallai theorem that has been pointed out by Erdős.) This conjecture remains open. When restricted to metric spaces induced by connected graphs, it has a possible strengthening: In the set of n-vertex connected graphs where no line consists of all n vertices, the number of distinct lines is maximized by a complete multipartite graph. Finally, a 3-uniform hypergraph H is called metric if there is a metric space where one of u, v, w lies between the other two if and only if $\{u, v, w\}$ is a hyperedge of H. How difficult is recognition of metric hypergraphs?